Problem Sheet 7

Problem 1

Show that the following two characterizations of lattices $\Lambda \subseteq \mathbb{R}^n$ are equivalent.

(a) Λ is a finitely generated subgroup such that the natural map $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \longrightarrow \mathbb{R}^n$ is an isomorphism.

(b) Λ is a discrete subgroup such that $\mathbb{R}^n/\Lambda$ is compact.

Problem 2

Let $K \subseteq \mathbb{R}$ be a real-quadratic field with discriminant D. The fundamental unit of K is defined to be the unique $\omega > 1$ such that $U_K = \{\pm 1\} \times \omega^{\mathbb{Z}}$.

(a) Let $u \in U_K$ with $u > 1$ and set $\epsilon := N_{K/\mathbb{Q}}(u) \in \{\pm 1\}$. Show that

$$u \geq \frac{(\sqrt{D} + \sqrt{D + 4\epsilon})}{2}.$$

Hint: Use $|\mathrm{Disc}_{K/\mathbb{Q}}(1, u)| \geq D$.

(b) Show that if there is a prime $p \mid D$ with $p \equiv 3 \mod 4$, then there is no $u \in U_K$ with $N_{K/\mathbb{Q}}(u) = -1$.

(c) Find the fundamental units of $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$.

Problem 3

Let $K = \mathbb{Q}(\zeta_p)$ with p an odd prime and set $K^+ := K(\zeta_p + \zeta_p^{-1})$. As usual, U_K and U_{K^+} denote the integral units of K and K^+.

(a) For $u \in U_K$, show that $u/\bar{u}$ is a root of unity.

(b) Show further that $u/\bar{u} = \zeta_p^k$ for some $k \in \mathbb{Z}$ (and not $u/\bar{u} = -\zeta_p^k$ for some k.)

(c) Conclude that $U_K = U_{K^+} \times \langle \zeta_p \rangle$.

Problem 4

Let $\zeta_5 = e^{2\pi i/5}$ and $u = -(\zeta_5^2 + \zeta_5^3)$.

(a) Show that u is quadratic over $\mathbb{Q}$ and determin the field $\mathbb{Q}(u)$.

(b) Prove that the units of $\mathbb{Q}(\zeta_5)$ are give by $\pm\zeta_5^k(1 + \zeta_5)^h$ with $0 \leq k \leq 4$ and $h \in \mathbb{Z}$.

(c) Determine the regulator of $\mathbb{Q}(\zeta_5)$.